NORTH SYDNEY GIRLS HIGH SCHOOL

2010
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions
- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page
- All necessary working should be shown in every question

Total Marks – 120
Attempt Questions 1–8
All questions are of equal value

At the end of the examination, place your solution booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

NAME:____________________________ **TEACHER:**________________________

NUMBER:________________________

QUESTION	MARK
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
8	/15
TOTAL	/120

BLANK PAGE

Please turn over

Total Marks – 120
Attempt Questions 1–8
All questions are of equal value

Begin each question in a NEW BOOKLET.

Question 1 (15 marks) **Marks**

(a) Find $\int \frac{\cos^2 x}{1-\sin x}\,dx$. **2**

(b) Use the method of partial fractions to find $\int \frac{1}{x^2+x}\,dx$. **3**

(c) (i) Use the table of standard integrals to find **1**

$$\int \frac{dx}{\sqrt{4x^2-1}}.$$

(ii) Is the following statement true or false? Justify your answer. **1**

$$\int_{-1}^{1} \frac{dx}{\sqrt{4x^2-1}} = 2\int_{0}^{1} \frac{dx}{\sqrt{4x^2-1}}.$$

(d) By using the substitution, $t = \tan\frac{\theta}{2}$, evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\sin\theta}.$$

(e) (i) Show that $\int_0^1 x^n e^{-x}\,dx = nI_{n-1} - \frac{1}{e}$ where $I_n = \int_0^1 x^n e^{-x}\,dx$. **3**

(ii) Hence deduce that $\int_0^1 x^3 e^{-x}\,dx = 6 - \frac{16}{e}$. **2**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given the complex number $z = -7 + 2i$, find

(i) $\overline{z}$ **1**

(ii) $\arg \overline{z}$ giving your answer to one decimal place. **1**

(iii) $\arg iz + \arg \overline{iz}$ **2**

(b) By using the modulus-argument form of a complex number, evaluate **2**

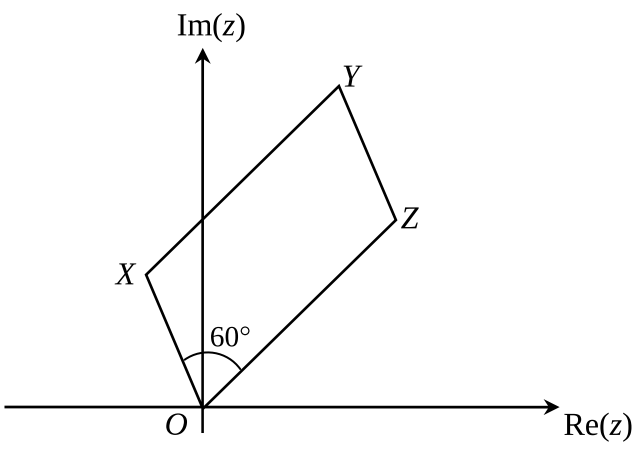
$$\left(1 - \sqrt{3}i\right)^9$$

(c) On the Argand diagram, sketch the region described by **3**

$$|z| < 2 \quad \text{and} \quad \frac{2\pi}{3} \le \arg z \le \frac{5\pi}{6}$$

(d) In the diagram below $OXYZ$ is a parallelogram with $OX = \frac{1}{2}OZ$

The point X represents the complex number $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$



If $\angle XOZ = 60°$, what complex number does Z represent? **2**

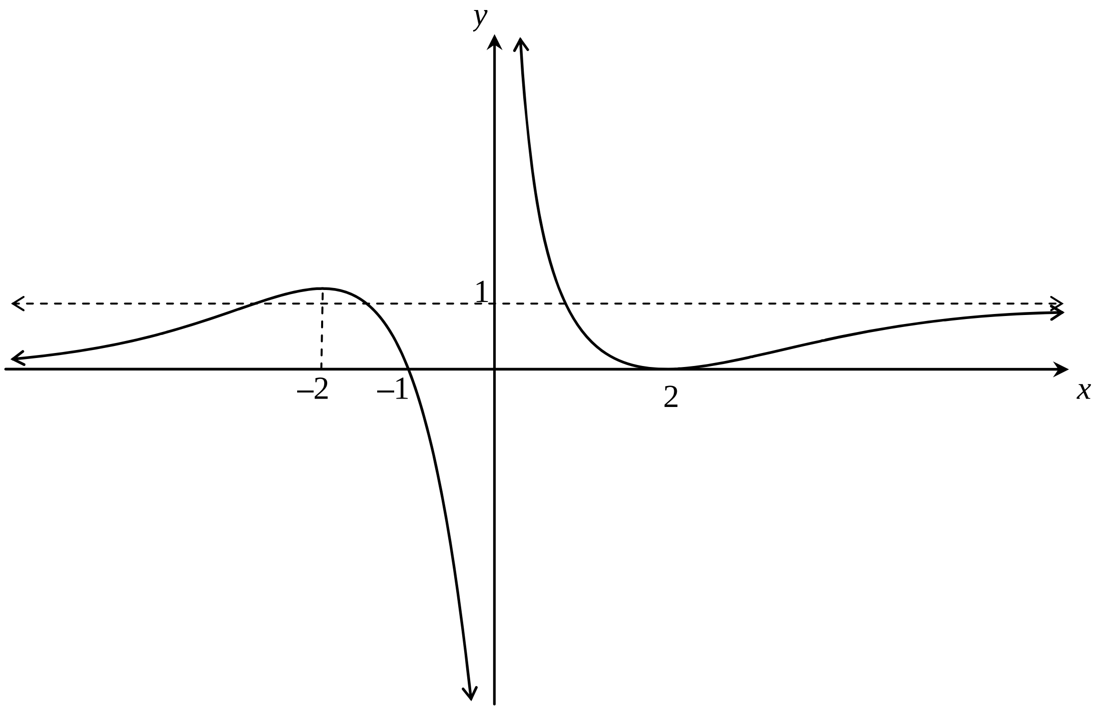
(e) Given that $z = \cos\theta + i\sin\theta$

(i) Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$. **1**

(ii) Hence, or otherwise, solve the equation $2z^4 - z^3 + 3z^2 - z + 2 = 0$. **3**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The graph of $y = f(x)$ is displayed below. The lines $y = 1$, $x = 0$ and $y = 0$ are asymptotes.



Using the separate page of graphs provided, sketch each of the graphs below. In each case, clearly label any maxima or minima, intercepts and the equations of any asymptotes.

(i) $y = f\left(|x|\right)$ **2**

(ii) $y^2 = f\left(x\right)$ **2**

(iii) $y = e^{f(x)}$ **2**

(iv) $y = \sin^{-1}\left[f\left(x\right)\right]$ **2**

(b) (i) On the same set of axes, sketch the graphs of $y = e^x$ and $y = \sqrt[3]{x}$. **2**

(ii) On another set of axes sketch, using the graphs in (i), the graph of **3**

$y = \sqrt[3]{x}e^x$.

(iii) Use the last sketch to determine the values of k for which the equation **2**

$\sqrt[3]{x} = \dfrac{kx+2}{e^x}$ has exactly one solution.

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the equation of the conic below

$$\frac{x^2}{29-\lambda} - \frac{y^2}{4-\lambda} = 1$$

(i) Find the values of λ for which this conic defines an ellipse. **2**

(ii) If the equation represents an hyperbola, show that the focus of the hyperbola is independent of λ. **2**

(iii) Sketch the conic defined by $\lambda = 13$. **3**

(b) A sequence $\mu_1, \mu_2, \mu_3 \ldots \mu_n$ is such that any three consecutive terms are related by the equation $\mu_{n+3} = 6\mu_{n+2} - 5\mu_{n+1}$. It is given that $\mu_1 = 2$ and $\mu_2 = 6$. **3**

Use mathematical induction to prove that $\mu_n = 5^{n-1} + 1$.

(c) The roots of the equation $x^3 + 2x - 8 = 0$ are α, β and γ.
Find the polynomial equation whose roots are given by

(i) $1-\alpha$, $1-\beta$ and $1-\gamma$. **3**

(ii) $\dfrac{\alpha+\beta}{\gamma}, \dfrac{\beta+\gamma}{\alpha}$ and $\dfrac{\alpha+\gamma}{\beta}$. **2**

[In part (ii) consider the relationship between the coefficients and the roots]

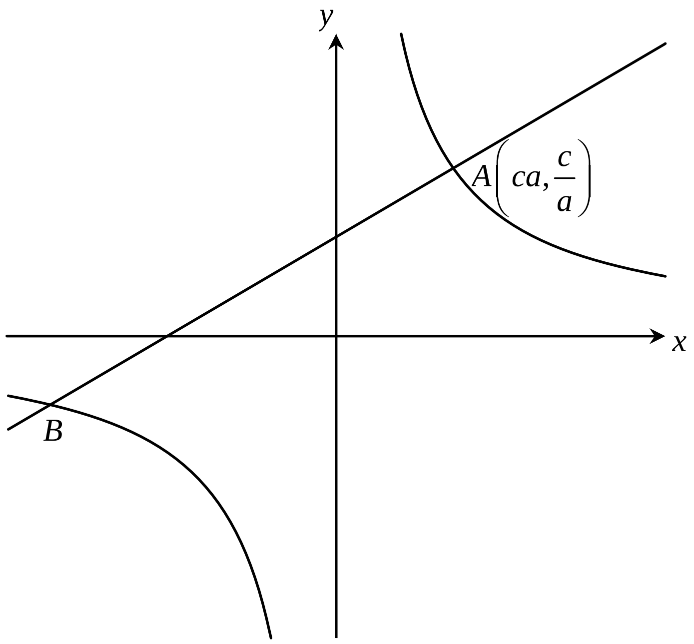
Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Five lines are drawn in a plane. No two lines are parallel and no three lines are concurrent.

(i) Show that there are 10 points of intersection giving a reason for your answer. **1**

(ii) If three of the points are chosen at random, find the probability that they all lie on one of the given lines **2**

(b) Consider the rectangular hyperbola with equation $xy = c^2$ with points A and B which is shown below. The normal through A on the hyperbola meets the other branch at B.



(i) Show that the equation of the normal is given by **2**

$$y = a^2 x + \frac{c}{a}\left(1 - a^4\right)$$

(ii) If B has co-ordinates $\left(cb, \frac{c}{b}\right)$, show that $b = -\frac{1}{a^3}$ **3**

(iii) If this hyperbola is rotated clockwise through 45°, show that the equation becomes $x^2 - y^2 = 2c^2$. **3**

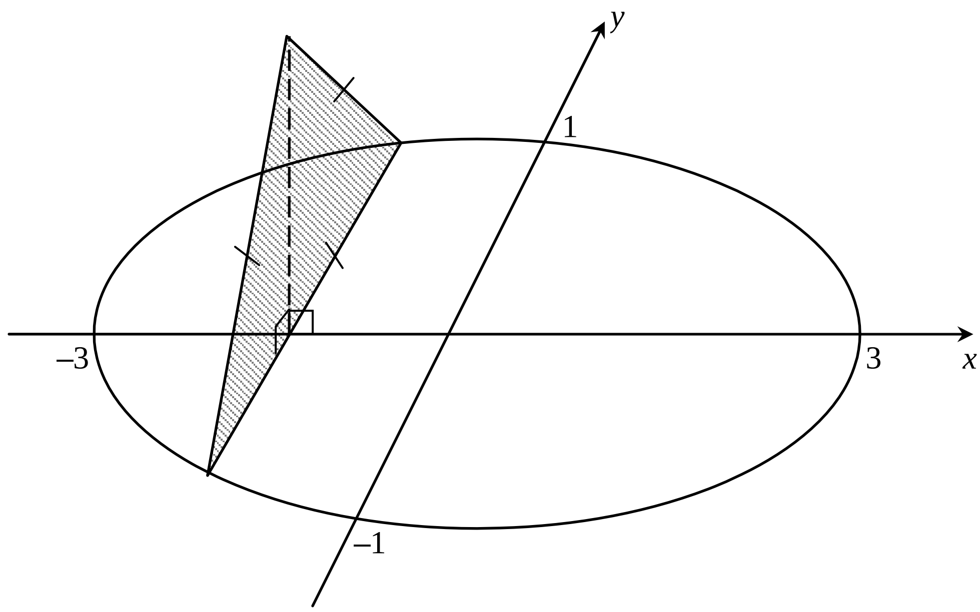
(c) If x and y are positive numbers such that $x + y = 1$, prove that

(i) $\frac{1}{x} + \frac{1}{y} \geq 4$ **2**

(ii) $x^2 + y^2 \geq \frac{1}{2}$ **2**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A solid shape has an elliptical base on the *xy*-plane as shown below.
Sections of the solid taken perpendicular to the *x*-axis are equilateral triangles.
The major and minor axes of the ellipse are of lengths 6 metres and 2 metres respectively.



(i) Write down the equation of the ellipse. **1**

(ii) Show that the volume ΔV of a slice taken at $x = d$ is given by **2**

$$\Delta V \doteqdot \frac{\sqrt{3}\left(9-d^2\right)}{9}\Delta x$$

(iii) Find the volume of this solid. **3**

(b) (i) Use graphs, or otherwise, to show that $\log_e(1+x) < x$ for $x > 0$. **2**

(ii) Sketch the graphs of $y = \frac{x}{1+x}$ and $y = \log_e(1+x)$ on the same set of axes and explain why $\frac{x}{1+x} < \log_e(1+x)$ for $x > 0$. **3**

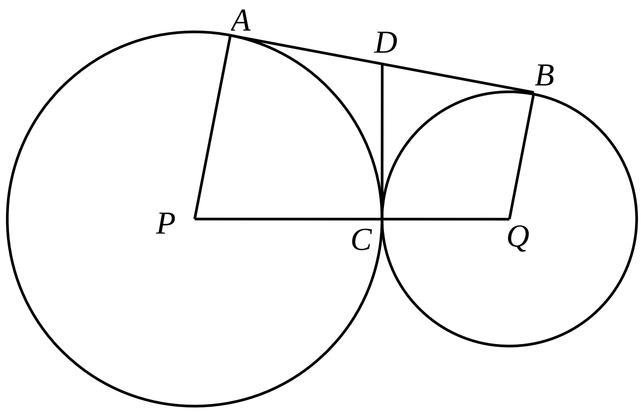
(iii) Using the inequalities in part (i) and (ii), show that **4**

$$\frac{\pi}{8} - \frac{1}{4}\log_e 2 < \int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx < \log_e\sqrt{2}$$

You may assume that $\displaystyle\int \frac{x\,dx}{(1+x)\left(1+x^2\right)} = \frac{\tan^{-1}x}{2} + \frac{\ln\left(x^2+1\right)}{4} - \frac{\ln(x+1)}{2}.$

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the diagram, PCQ is a straight line joining P and Q, the centres of the circles. AB and DC are common tangents.



(i) Copy the diagram into your answer booklet.

(ii) Explain why $PADC$ and $CDBQ$ are cyclic quadrilaterals. **2**

(iii) Show that ΔADC is similar to ΔBQC. **2**

(iv) Show that PD is parallel to CB. **2**

(b) (i) Sketch the region which is enclosed by the curve $y = 8x - x^2$ and the lines $x = 2$ and $x = 4$. **1**

(ii) This region is rotated about the y-axis to generate a solid. Represent this situation on a number plane and use the method of cylindrical shells to find the volume of the solid formed. **3**

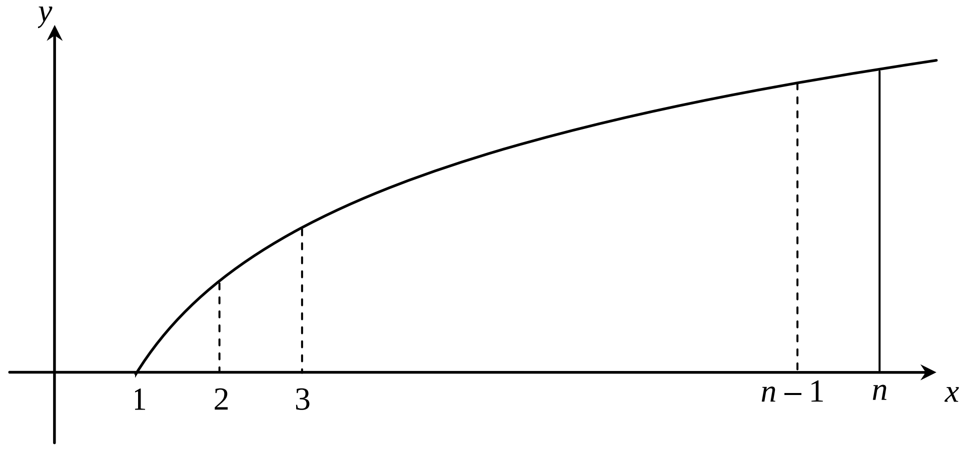
(c) (i) Show that the function $G(x)$ where $G(x) = \frac{1}{2}\left[f(x) + f(-x)\right]$ is even and that the function $H(x)$ where $H(x) = \frac{1}{2}\left[f(x) - f(-x)\right]$ is odd. **2**

(ii) Deduce that the function $f(x)$ can be written as the sum of an even function and an odd function. **1**

(iii) If $f(x) = 2^x + \tan x$, express $f(x)$ as the sum of an even function and an odd function. **2**

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The sketch below shows the graph of $y = \log_e x$ for $x \geq 1$.



(i) Explain why this curve is always concave downwards. **1**

(ii) Show that the area under $y = \log_e x$ from $x = 1$ to $x = n$ is given by $n \ln n - n + 1$ **1**

(iii) By adding the areas of the trapezia from $x = 1$ to $x = n$, show that **3**

$$n! < \frac{en^{n+\frac{1}{2}}}{e^n}$$

(b) $P(x)$ is a polynomial of degree at least 2, such that $P'(a) = 0$. Show that when $P(x)$ is divided by $(x-a)^2$ the remainder is $P(a)$. **3**

(c) Use mathematical induction to show that **3**

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1 \text{ for } n \geq 1$$

(d) (i) Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot \theta$. **2**

(ii) Hence find the value of $\cot \frac{\pi}{8}$ and $\cot \frac{\pi}{12}$ and show that **2**

$$\operatorname{cosec} \frac{2\pi}{15} + \operatorname{cosec} \frac{4\pi}{15} + \operatorname{cosec} \frac{8\pi}{15} + \operatorname{cosec} \frac{16\pi}{15} = 0$$

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

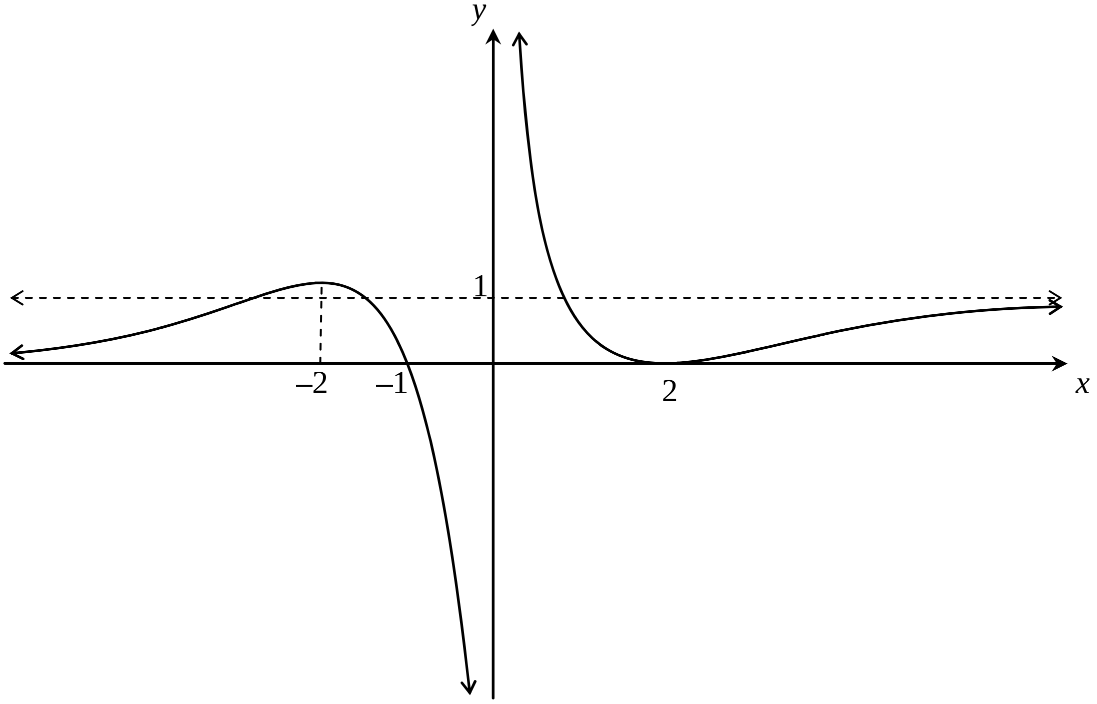
NOTE : $\ln x = \log_e x, \quad x > 0$

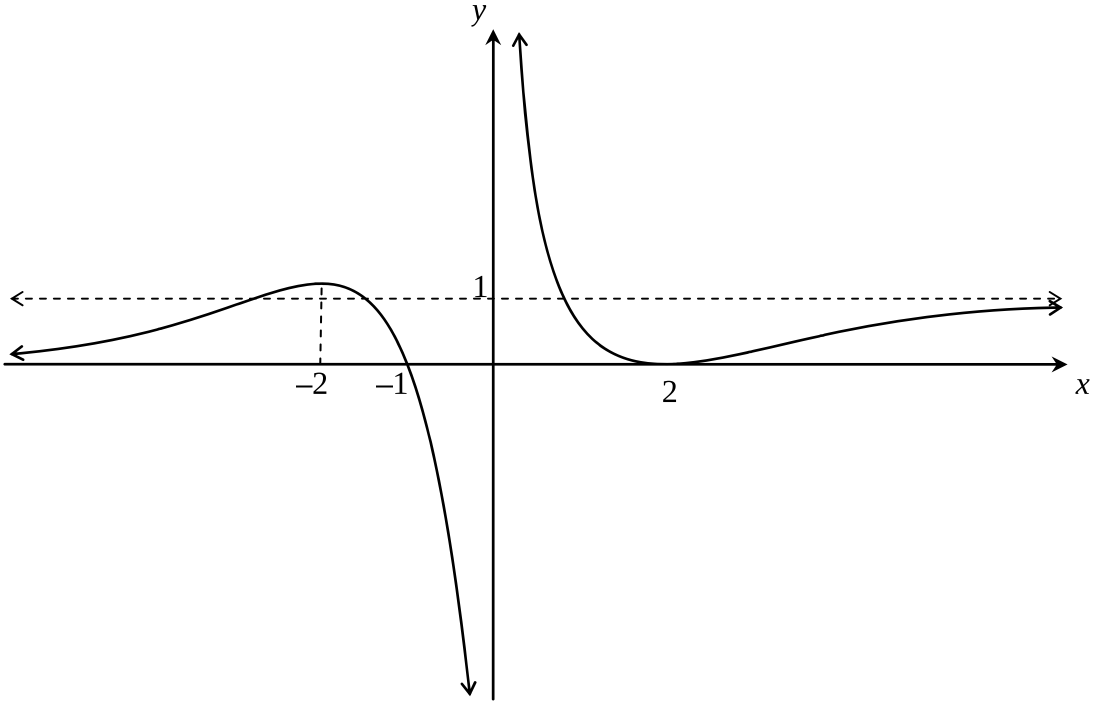
BLANK PAGE

Please turn over

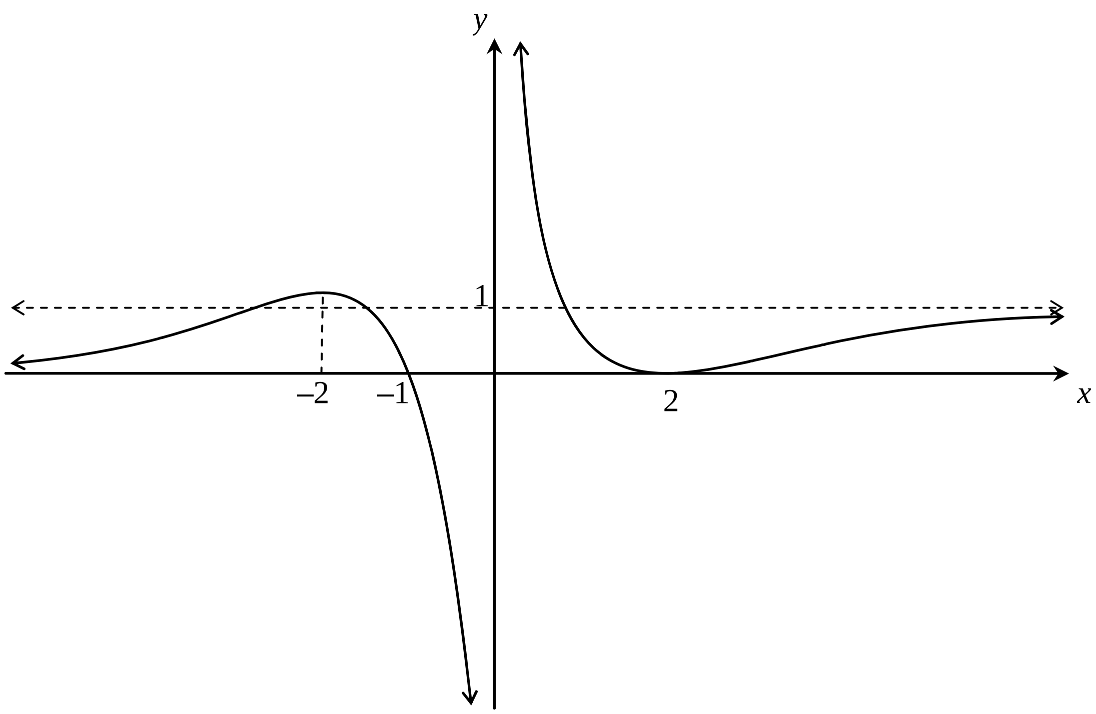
Student Number:____________________________

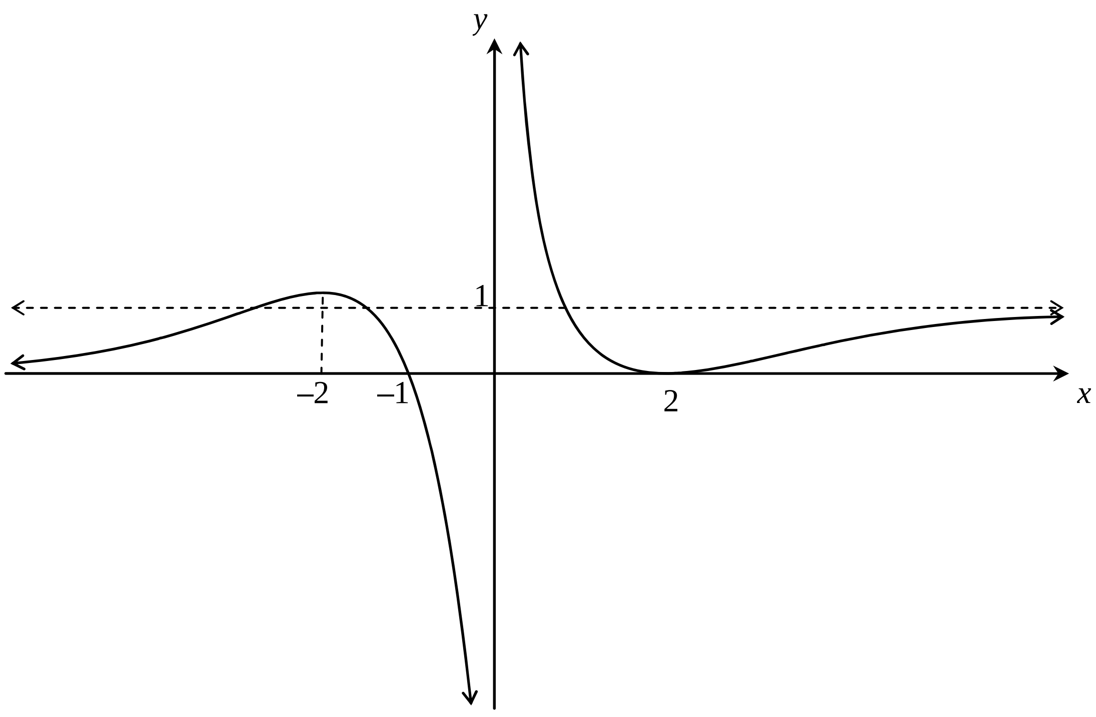
Use this page for your answers to Question 3a)





Insert this page in your booklet for Question 3





Insert this page in your booklet for Question 3

Q1

a) $I = \int \frac{\cos^2 x \, dx}{1 - \sin x}$

$= \int \frac{1 - \sin^2 x}{1 - \sin x} dx$

$= \int (1 + \sin x) dx$

$= x - \cos x + C$

b) $I = \int \frac{1}{x(x+1)}$

Let $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$1 = A(x+1) + Bx$

Let $x = -1$

$1 = -B$

$B = -1$

Let $x = 0$

$1 = A$

$I = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= \ln x - \ln(x+1) + C$

$= \ln \frac{x}{x+1} + C$

(c) (i) $I = \int \frac{dx}{\sqrt{4x^2 - 1}}$

$= \int \frac{dx}{2\sqrt{x^2 - \frac{1}{4}}}$

$= \frac{1}{2} \int \frac{dx}{\sqrt{x^2 - \frac{1}{4}}}$

$= \frac{1}{2} \ln\left(x + \sqrt{x^2 - \frac{1}{4}}\right) + C$

(ii) False as $4x^2 - 1 > 0$

ie $x < -\frac{1}{2}$ or $x > \frac{1}{2}$

and the limits of integral are between -1 and 1

d) If $t = \tan \frac{\theta}{2}$

$\theta = 2 \tan^{-1} t$

$d\theta = \frac{2dt}{1+t^2}$

$\theta = 0, \theta = \frac{\pi}{2}$

$t = 0 \quad t = 1$

$I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{2 + \sin\theta}$

$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{2 + \frac{2t}{1+t^2}}$

$= \int_0^1 \frac{2dt}{2(1+t^2) + 2t}$

$= \int \frac{dt}{t^2 + t + 1}$

$= \int \frac{dt}{t^2 + t + \frac{1}{4} + \frac{3}{4}}$

$= \int_0^1 \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{2\left(t + \frac{1}{2}\right)}{\sqrt{3}}\right]_0^1$

$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \sqrt{3} - \tan^{-1} \frac{1}{\sqrt{3}}\right]$

$= \frac{2}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6}\right)$

$= \frac{\pi}{3\sqrt{3}}$

Q1 Q2

a) (i)

$$I = \int_0^1 x^n e^{-x}\,dx$$

$$= \int_0^1 \left(\frac{d}{dx} - e^{-x}\right) x^n\,dx$$

$$= -\left[e^{-x}x^n\right]_0^1 + n\int_0^1 e^{-x}x^{n-1}\,dx$$

$$= -\left[\frac{1}{e} - 0\right] + n I_{n-1}$$

$$= n I_{n-1} - \frac{1}{e}$$

(ii)

$$I_3 = 3I_2 - \frac{1}{e}$$

$$I_0 = \int_0^1 e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_0^1$$

$$= \left[-e^{-1} + 1\right]$$

$$I_1 = I_0 - \frac{1}{e}$$

$$= -e^{-1} + 1 - \frac{1}{e}$$

$$= 1 - \frac{2}{e}$$

$$I_2 = 2I_1 - \frac{1}{e}$$

$$= 2\left(1 - \frac{2}{e}\right) - \frac{1}{e}$$

$$= 2 - \frac{4}{e} - \frac{1}{e}$$

$$= 2 - \frac{5}{e}$$

$$I_3 = 3I_2 - \frac{1}{e}$$

$$= 3\left(2 - \frac{5}{e}\right) - \frac{1}{e}$$

$$= 6 - \frac{15}{e} - \frac{1}{e}$$

$$= 6 - \frac{16}{e}$$

$$8\cos^2\theta - 2\cos\theta - 1 = 0$$

$$(2\cos\theta - 1)(4\cos\theta + 1) = 0$$

$\cos\theta = \frac{1}{2}, -\frac{1}{4}$ $\quad z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ OR $-\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$

$\sin\theta = \pm\frac{\sqrt{3}}{2}$

Q2 a) (i) $-7 - 2i$

(ii) $\tan^{-1}\frac{2}{7} - \pi$

(iii) $\arg(iz \cdot i\bar{z}) = \arg(\cdot\, z\bar{z})$

$= \arg(\cdot\, 53)$

$= 0$

b) $1 - \sqrt{3}i = 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$

$$(1-\sqrt{3}i)^9 = \left[2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)\right]^9$$

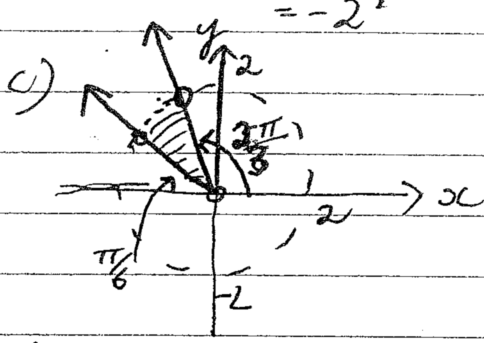
$$= 2^9\left(\mathrm{cis}(-3\pi)\right)$$

$$= 2^9\left(\cos(-3\pi) + i\sin(-3\pi)\right)$$

$$= 2^9(-1)$$

$$= -2^9$$

c)



d) $OZ = 2 \times OX \times \mathrm{cis}\left(-\frac{\pi}{3}\right)$

$$= 2 \times \mathrm{cis}\,\frac{2\pi}{3} \times \mathrm{cis}\left(-\frac{\pi}{3}\right)$$

$$= 2\,\mathrm{cis}\,\frac{\pi}{3}$$

$$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= 1 + i\sqrt{3}$$

e) (i) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$$= 2\cos n\theta$$

(ii) $2z^4 - z^3 + 3z^2 - z + 2 = 0$

$$2z^4 + 2 - z^3 - z + 3z^2 = 0$$

$$2(z^4 + 1) - (z^3 + z) + 3z^2 = 0 \quad \leftarrow (\div z^2)$$

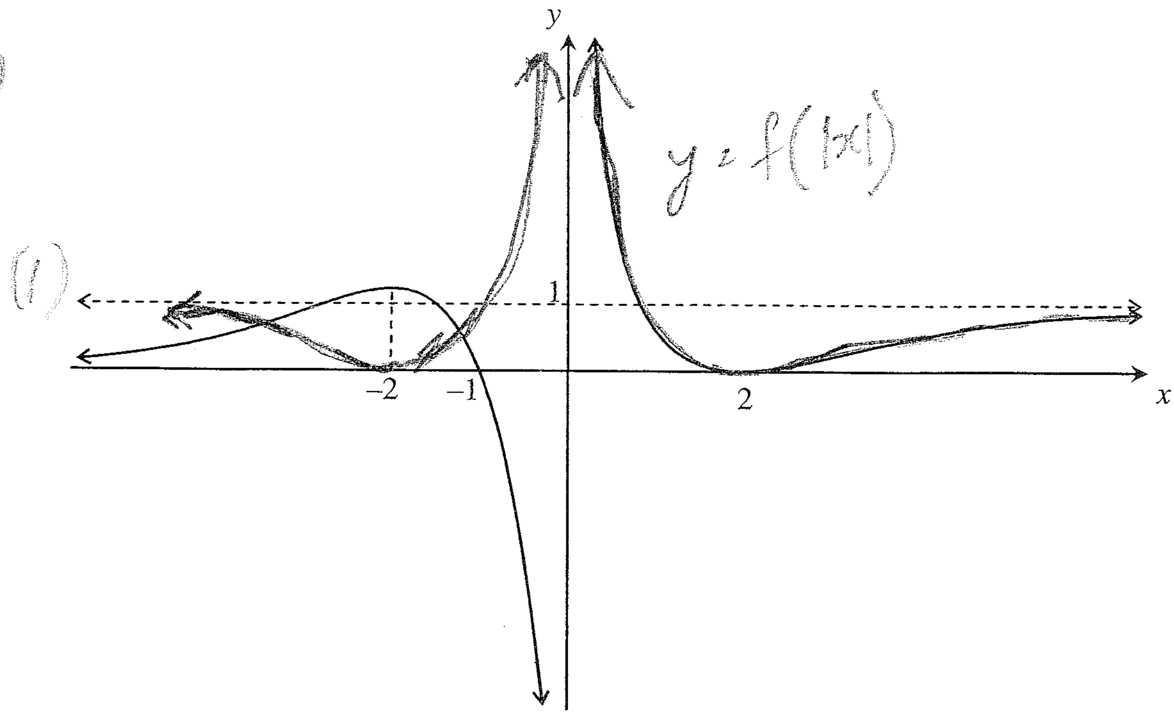
$$2\left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 3 = 0$$

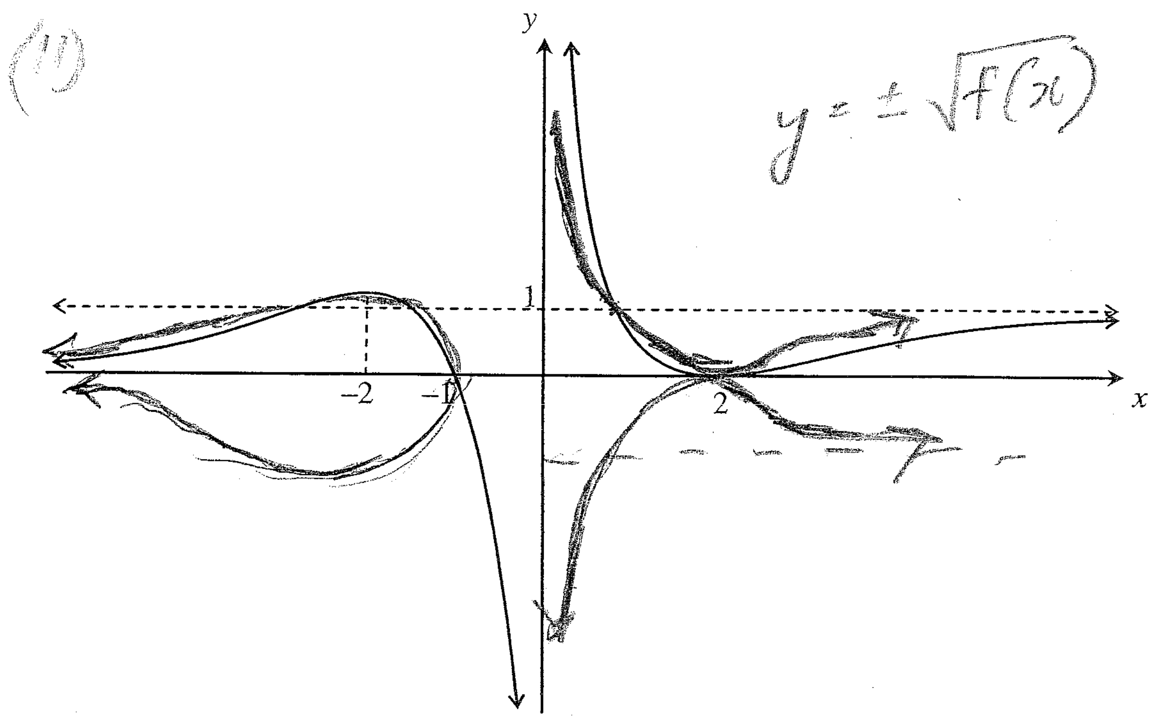
$$2(2\cos 2\theta - 2\cos\theta + 3) = 0$$

$$4\cos 2\theta - 2\cos\theta + 3 = 0$$

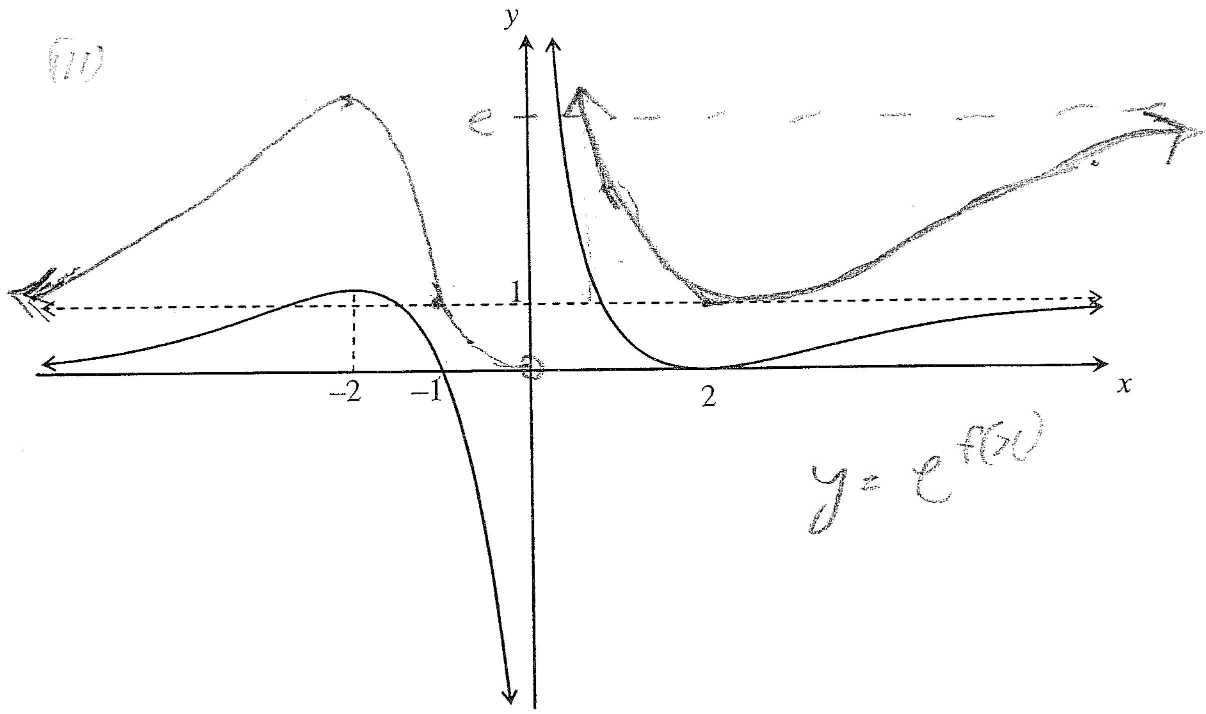
$$4(2\cos^2\theta - 1) - 2\cos\theta + 3 = 0$$

3(a)

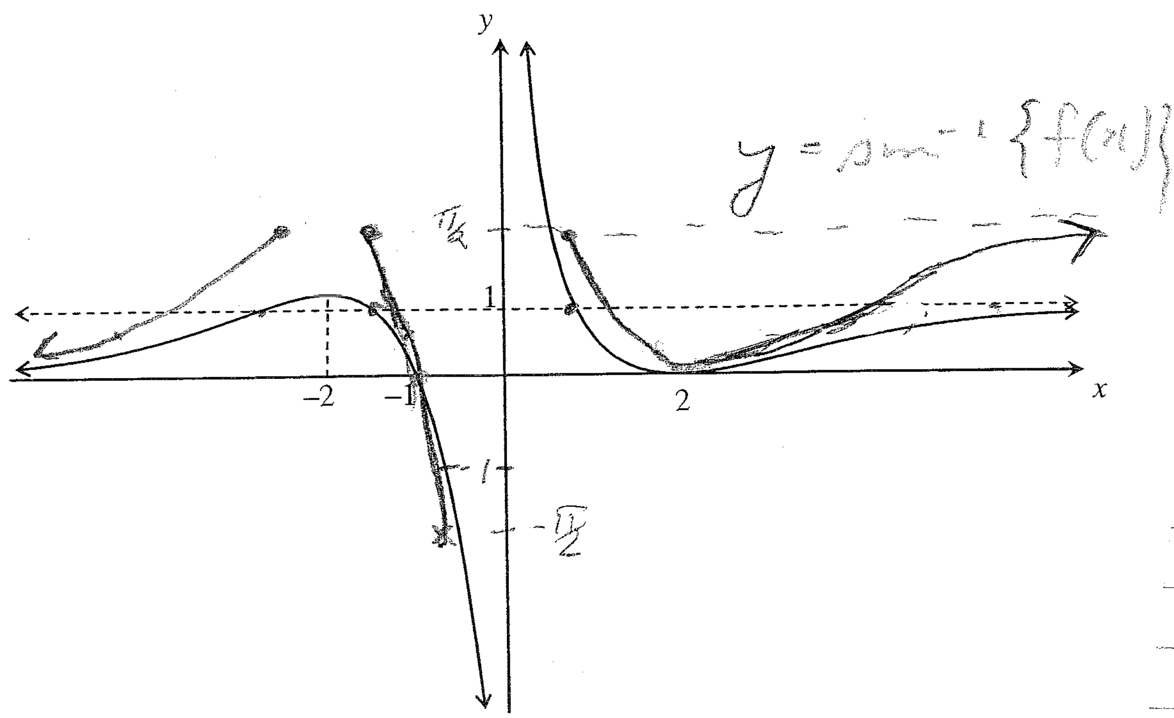




3(a)
(iii)



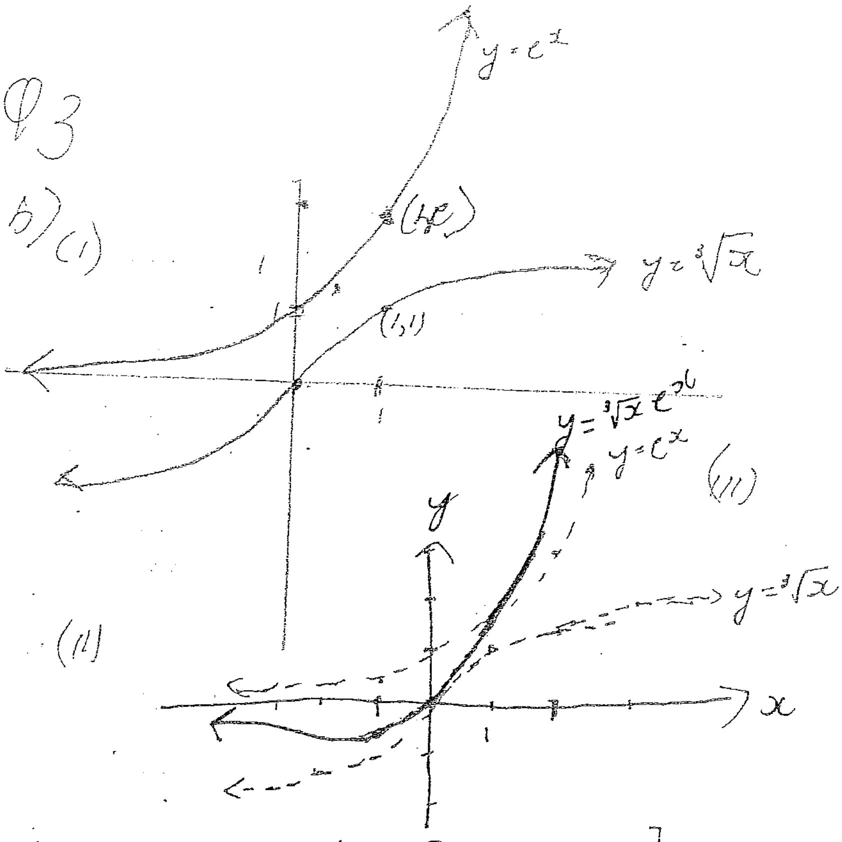
3a(iv)



Q3

b) (i)

(ii)



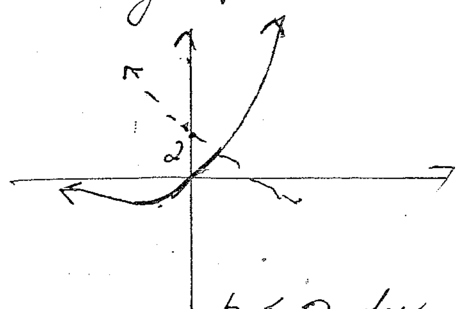
(iii) $\sqrt[3]{x} = \dfrac{kx+2}{e^x}$

$\sqrt[3]{x}\,e^x = kx+2$

derives from solving simultaneously

$y=\sqrt[3]{x}\,e^x$ and $y=kx+2$

$$\left(\begin{aligned} y &= x^{\frac{1}{3}}e^x \\ y' &= \tfrac{1}{3}x^{-\frac{2}{3}}e^x + x^{\frac{1}{3}}e^x \\ &= e^x\left(\frac{1}{3x^{\frac{2}{3}}} + \frac{x^{\frac{1}{3}}}{1}\right) \\ &= e^x\left(\frac{1+3x^2}{3x^{\frac{2}{3}}}\right) \end{aligned}\right)$$



$k \leq 0$ for exactly one solution

Question 4

a) (I) $\dfrac{x^2}{29-\lambda} - \dfrac{y^2}{4-\lambda} = 1$

We require

$29-\lambda > 0$ AND $4-\lambda < 0$

$\lambda > 4$

$\lambda < 29$

ie $4 < \lambda < 29$

(II) Now FOR HYPERBOLA

$a^2 - b^2 = a^2e^2$

$29-\lambda-(4-\lambda) = a^2e^2$

$29-4 = a^2e^2$

$a^2e^2 = 25$

$ae = \pm 5$

$S(\pm 5, 0)$

FOR ELLIPSE

$29-\lambda-(\lambda-4) = a^2e^2$

$33-2\lambda = a^2e^2$

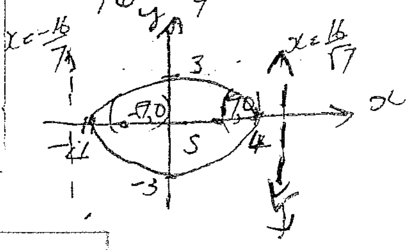
$ae = \sqrt{33-2\lambda}$

THIS ANOMALY WAS TAKEN INTO ACCOUNT IN THE MARKING

(III) $\lambda = 13$

$\dfrac{x^2}{16} - \dfrac{y^2}{-9} = 1$

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$



b) Step 1

Test $n_1 = 1$

$u_1 = 5^0 + 1$

$= 2$

$= u_1$ as give

$n = 2$

$u_2 = 5^1 + 1$

$= 6$

$= u_2$ as given

Step 2

Assume $u_k = 5^{k-1} + 1$ and

since $u_{k+3} = 6u_{k+2} - 5u_{k+1}$ (*)

also $u_{k+1} = 5^k + 1$

$u_{k+2} = 5^{k+1} + 1$

$u_{k+3} = 5^{k+2} + 1$

LHS of (*)

LHS $= u_{k+3}$

$= 5^{k+2} + 1$

RHS $= 6u_{k+2} - 5u_{k+1}$

$= 6(5^{k+1}+1) - 5(5^k+1)$

$= 6.5^{k+1} + 6 - 5^{k+1} - 5$

$= 5.5^{k+1} + 1$

$= 5^{k+2} + 1$

Step 3

Since the formula is true for $n = 1$, $n = 2$ and is true for $n = k+1$ if true for $n = k$ then true for all $n \geq 1$

$$x^3 + 2x - 8 = 0$$

4 (c) (i)

Let $X = 1 - x$

$\therefore x = 1 - X$

$x^3 + 2x - 8 = 0$

$(1-X)^3 + 2(1-X) - 8 = 0$

$(1-X)\{(1-X)^2 + 2\} - 8 = 0$

$(1-X)(X^2 - 2X + 3) - 8 = 0$

$X^2 - 2X + 3 - X^3 + 2X^2 - 3X - 8 = 0$

$-X^3 + 3X^2 - 5X - 5 = 0$

ie $x^3 - 3x^2 + 5x + 5 = 0$

(ii) $\alpha + \beta + \gamma = 0$

$\therefore \alpha + \beta = -\gamma$

$\beta + \gamma = -\alpha$

$\alpha + \gamma = -\beta$

$\frac{\alpha+\beta}{\gamma}, \frac{\beta+\gamma}{\alpha}$ and $\frac{\alpha+\gamma}{\beta}$

become

$\therefore -\frac{\gamma}{\gamma}, -\frac{\alpha}{\alpha}, -\frac{\beta}{\beta}$

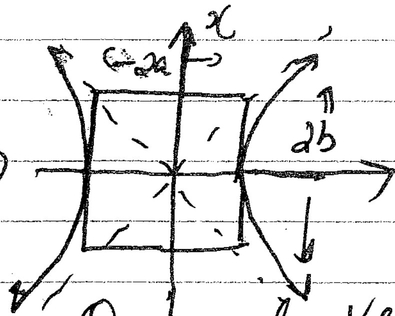
ie triple root of -1

ie $(x+1)^3 = 0$

Q5 a) (i) ${}^5C_2$

(ii) $\dfrac{5 \times {}^4C_3}{{}^{10}C_3} = \dfrac{20}{120} = \dfrac{1}{6}$

(iii)



Original vertices $(\pm c, \pm c)$

Distance from O is $\sqrt{2}c$

New vertices are $(\pm\sqrt{2}, 0) \Rightarrow a = \sqrt{2}c$

Since hyperbola is rectangular $b = \sqrt{2}c$

$\dfrac{x^2}{(\sqrt{2}c)^2} + \dfrac{y^2}{(\sqrt{2}c)^2} = \dfrac{x^2}{(\sqrt{2}c)^2} - \dfrac{y^2}{(\sqrt{2}c)^2} = 1$

ie $x^2 - y^2 = 2c^2$

b)

(i) $y = \dfrac{c^2}{x}$

$\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

at $x = ca$

$y' = -\dfrac{1}{a^2}$

$m_{NORMAL} = a^2$

$\therefore y - \dfrac{c}{a} = a^2(x - ca)$

$= a^2x - ca^3$

$y = a^2x + \dfrac{c}{a} - ca^3$

$= a^2x + \dfrac{c}{a}(1 - a^4)$ (1)

(ii) Solve (1) simultaneously with $y = \dfrac{c^2}{x}$

$\dfrac{c^2}{x} = a^2x + \dfrac{c}{a}(1 - a^4)$

$a^2x^2 + \dfrac{c}{a}(1 - a^4)x - c^2 = 0$

Let the roots be α, β.

$\alpha\beta = -\dfrac{c^2}{a}$

$\therefore$ roots are

cb, ca

$\therefore c^2ab = -\dfrac{c^2}{a}$

$b = -\dfrac{1}{a^2}$

c) $x + y = 1$

(i) $\dfrac{1}{x} + \dfrac{1}{y} \geq 4$

LHS $= \dfrac{1}{x} + \dfrac{1}{1-x}$

$= \dfrac{1}{x(1-x)}$

≥ 4

$=$ RHS

$x(1-x)$ has max value of $\frac{1}{4}$

$y = x(1-x)$

$V(\frac{1}{2}, \frac{1}{4})$

$\therefore$ least value of $\frac{1}{x(1-x)}$ is 4

(ii) $x^2 + y^2 \geq \frac{1}{2}$

From $\dfrac{1}{xy} \geq 4 \Rightarrow xy \leq \dfrac{1}{4}$

Consider $(x-y)^2 \geq 0$

$x^2 - 2xy + y^2 \geq 0$

$x^2 + y^2 \geq 2xy$

Since $xy \leq \frac{1}{4}$,

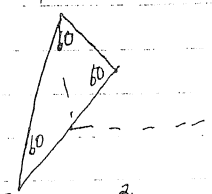
$x^2 + y^2 \geq 2 \times \frac{1}{4}$

$x^2 + y^2 \geq \frac{1}{2}$

6 a)

(i) $\frac{x^2}{9} + y^2 = 1$

(ii)



$y^2 = 1 - \frac{x^2}{9}$

$y^2 = \frac{9-x^2}{9}$

$y = \frac{\sqrt{9-x^2}}{3}$

$A = \frac{1}{2} \times 2\frac{\sqrt{9-x^2}}{3} \, 2\frac{\sqrt{9-x^2}}{3} \sin 60$

$= \frac{1}{2} \cdot \frac{4(9-x^2)}{9} \, \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}(9-x^2)}{9}$

$\Delta V = \frac{\sqrt{3}(9-x^2)}{9} \Delta x$

b) $V = \sqrt{3} \int_{-3}^{3} \frac{(9-x^2)}{9} \, dx$

$= \frac{2\sqrt{3}}{9} \int_{0}^{3} (9-x^2) \, dx$

$= \frac{2\sqrt{3}}{9} \left[9x - \frac{x^3}{3} \right]_0^3$

$= \frac{2\sqrt{3}}{9} \left[(27-9) - 0 \right]$

$= \frac{2\sqrt{3}}{9} \times 18$

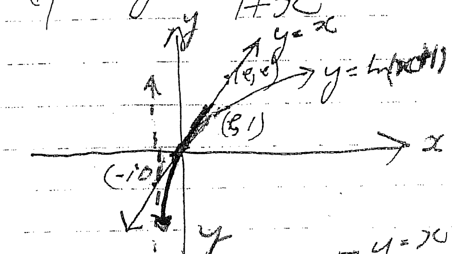
$= 4\sqrt{3}$

$4\sqrt{3}$ cubic units

Q6 b)

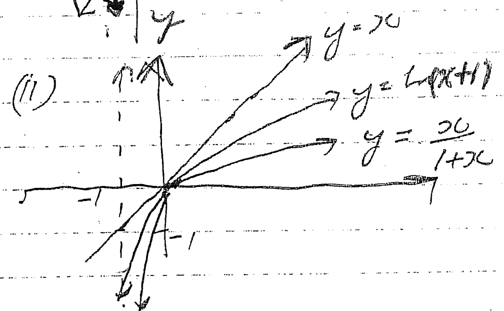
(i) $y = \ln(1+x)$

$y' = \frac{1}{1+x}$



at $x = 0$

(ii)



$y = \frac{x}{1+x}$ $\qquad$ $y = \ln(1+x)$

$y' = \frac{1}{(1+x)^2}$ $\qquad$ $y' = \frac{1}{1+x}$

Gradient function of $y = \frac{1}{(1+x)^2}$ is less than $\frac{1}{1+x}$ for $x > 0$

Both are tangential to $y = x$ at $x = 0$

so because $\frac{1}{(1+x)^2} < \frac{1}{1+x}$ for $x > 0$

then the graph of $y = \frac{x}{1+x}$ is below the graph of $y = \log_e(1+x)$ for $x \geq 0$

(16b)

(III) From (I) & (II)

$$\frac{x}{1+x} < \ln(1+x) < x \quad x>0$$

Hence $\dfrac{x}{(1+x)(1+x^2)} < \dfrac{\ln(1+x)}{1+x^2} < \dfrac{x}{1+x^2} \quad x>0$

$$\int_0^1 \frac{x\,dx}{(1+x)(1+x^2)} < \int_0^1 \frac{\ln(1+x)}{1+x^2}dx < \int_0^1 \frac{x}{1+x^2}dx \quad x>0$$

Now $\displaystyle\int_0^1 \frac{x}{1+x^2} = \frac{1}{2}\Big[\ln(x^2+1)\Big]_0^1$

$$= \frac{1}{2}\ln 2$$

and $\displaystyle\int_0^1 \frac{x}{(1+x)(1+x)^2} = \int_0^1 \left(-\frac{1}{2(x+1)} + \frac{1}{2(x^2+1)}\right)dx$

Integral is given in the question

$$= \Big[-\tfrac{1}{2}\ln(1+x)\Big]_0^1 + \Big[\tfrac{1}{4}\ln(x^2+1)\Big]_0^1 + \tfrac{1}{2}\Big[\tan^{-1}x\Big]_0^1$$

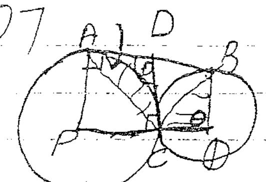
$$= \tfrac{1}{2}\ln 2 + \tfrac{1}{4}\ln 2 + \tfrac{1}{2}\tan^{-1}1$$

$$= \frac{\pi}{8} - \frac{1}{4}\ln 2$$

Hence $\dfrac{\pi}{8} - \dfrac{1}{4}\ln 2 < \displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}dx < \frac{1}{2}\ln 2 \quad x>0$

(90-θ/2)

Q7
2) a)



(i) ∠PAD = ∠DCP = 90° [Radius is perpendicular to tangent at point of contact]

∴ PADC is cyclic (Opposite angles supplementary)

(ii) Let ∠ADC = θ

∴ ∠BΦC = θ (opp angles of cyclic quadrilateral)

DA = DC equal tangents from external point

∴ △ ADC is isosceles

∴ ∠DAC = ∠DCA = (90 − θ/2)°

BΦ = CΦ equal radii

∴ △ ADC ||| △ CΦB [isosceles triangles equal angles]

(iii) From (i)

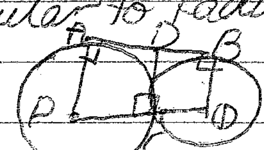
∠APC = (180 − θ)° (opp angles of cyclic PADC)

and ∠PDC = (90 − θ/2)° as PD bisects ∠APC

= ∠BCΦ

∴ PD ∥ CB corresponding angles equal

∠PAD = 90° [Tangent AB is perpendicular to radius AP]



∠DBΦ = 90° [Reason as above]

∠DCP = 90° [Reason as above]

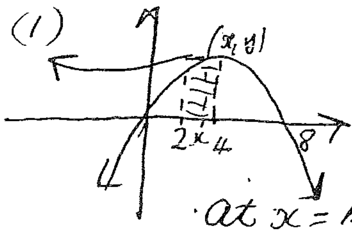
∠BDC = 90° [Reason as above]

∴ ∠PAD = ∠DCP = 90° ⟹ OPPOSITE ANGLES OF QUADRILATERALS ARE SUPPLEMENTARY

⟹ CYCLIC QUADRILATERALS

(i) 7b

(i)



at $x = k$

$\Delta V = \pi(k^2 - (k - \Delta k)^2)y$

$= \pi(2k\Delta k - (\Delta k)^2)y$ Let $\Delta k^2 = 0$

as $\Delta k \to 0$ or $\Delta x \to 0$

$V = 2\pi\int_2^4 x\,y\,dx$

$= 2\pi\int_2^4 x(8x - x^2)\,dx$

$= 2\pi\int_2^4 (8x^2 - x^3)\,dx$

$= 2\pi\left[\frac{8x^3}{3} - \frac{x^4}{4}\right]_2^4$

$= 2\pi\left[\left(\frac{512}{3} - 64\right) - \left(\frac{64}{3} - 4\right)\right]$

$= 2\pi\left[\frac{268}{3}\right]$

$= \frac{536}{3}\pi$ ie $\frac{536\pi}{3}$ cubic units

Q7 (i) If $G(x) = \frac{1}{2}[f(x)+f(-x)]$

$G(-x) = \frac{1}{2}[f(-x)+f(x)]$

$= G(x)$ ∴ even

If $H(x) = \frac{1}{2}[f(x)-f(-x)]$

the $H(-x) = \frac{1}{2}[f(-x)-f(x)]$

$= -\frac{1}{2}[f(x)-f(-x)]$

$= -H(x)$

(ii) So $f(x) = \frac{1}{2}[f(x)+f(-x)] + \frac{1}{2}[f(x)-f(-x)]$

$= f(x)$

$= G(x) + H(x)$

(iii)

$f(x) = 2^x + \tan x$

$G(x) = \frac{1}{2}[2^x + \tan x + 2^{-x} - \tan x]$

$= \frac{1}{2}[2^x + \frac{1}{2^x}]$

$H(x) = \frac{1}{2}[2^x + \tan x - (2^{-x} - \tan x)]$

$= \frac{1}{2}[2^x - 2^{-x} + 2\tan x]$ ∴ $\cancel{f(x) = \frac{1}{2}[2^x + 2^{-x}] [2^x + 2^{-x}]}$

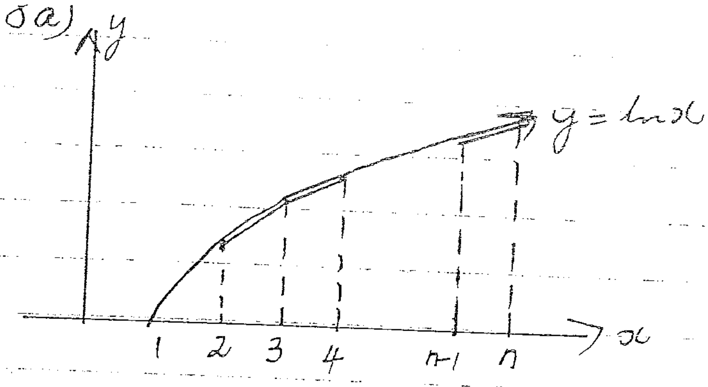
∴ $f(x) = G(x) + H(x)$ from above

$= \frac{1}{2}[2^x + \frac{1}{2^x} + 2^x - \frac{1}{2^x} + 2\tan x]$

$= \frac{1}{2}[2^{x+1} + 2\tan x]$

$= 2^x + \tan x$

5a)



(i) $y' = \frac{1}{x}$

$y'' = -\frac{1}{x^2}$

< 0 for $x > 0$

(ii) $I = \int_1^n \ln x \, dx$

$= \int_1^n \left(\frac{d}{dx} x\right) \ln x \, dx$

$= \left[x \ln x\right]_1^n - \int_1^n dx$

$= (n \ln n - 0) - (n - 1)$

$= n \ln n - n + 1$

(iii) Using trapezia

$A \doteq \frac{1}{2}\left[\ln 1 + \ln n + 2\left(\ln 2 + \ln 3 + \cdot + \ln(n-1)\right)\right]$

$= \frac{1}{2}\left[\ln n + 2 \ln (n-1)!\right]$

$= \frac{1}{2} \ln n + \ln (n-1)!$

$\therefore n \ln n - n + 1 > \frac{1}{2} \ln n + \ln (n-1)!$

$= \frac{1}{2} \ln n + \ln\left(\frac{n!}{n}\right)$

$= \frac{1}{2} \ln n + \ln n! - \ln n$

$= \ln n! - \frac{1}{2} \ln n$

ie $n \ln n - n + 1 > \ln n! - \frac{1}{2} \ln n$

$(n + \frac{1}{2}) \ln n - n + 1 > \ln n!$

$\therefore \ln n^{n+\frac{1}{2}} - n + 1 > \ln n!$

$\therefore e^{\ln n^{n+\frac{1}{2}} - n + 1} > e^{\ln n!}$

$n^{n+\frac{1}{2}} e^{-n} \cdot e > n!$

$\therefore n! < \frac{e n^{n+\frac{1}{2}}}{e^n}$

8b)

Divisor is $(x-a)^2$

so remainder is $cx+d$

$P(x) = (x-a)^2 Q(x) + cx+d$

$P'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x) + c$

$P'(a) = c$ Put $P'(a) = 0 \Rightarrow c = 0$

$\therefore P(x) = (x-a)^2 Q(x) + d$

$P(a) = d$

So the remainder $cx+d$

is $0x + P(a)$ ie $P(a)$ is the remainder

8 (c) (i)

$1\times1! + 2\times2! + 3\times3! + \cdots + n\times n! = (n+1)! - 1$

Step 1

$T_1 = 1$

$S_1 = (1+1)! - 1$
$= 1$

Step 2
assume true for $n = k$

ie $1\times1! + 2\times2! + 3\times3! + \cdots + k.k! = (k+1)! - 1$

Show that

$1\times1! + 2\times2! + 3\times3! + \cdots + k.k! + (k+1)(k+1)! = (k+2)! - 1$

$LHS = (1\times1! + 2\times2! + 3\times3! + \cdots + k.k!) + (k+1)(k+1)!$
$= (k+1)! - 1 + (k+1)(k+1)!$
$= (k+1)![1 + k + 1] - 1$
$= (k+2)(k+1)! - 1$
$= (k+2)! - 1$

Step 3 Since true for $n = 1$ and true for $n = k+1$ if true for $n = k$ then true for $n \geq 1$ and so on

(d) $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$

$LHS = \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}$
$= \frac{1+\cos 2\theta}{\sin 2\theta}$
$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta}$
$= \frac{\cos\theta}{\sin\theta}$
$= \cot\theta$

(iii) $LHS = \operatorname{cosec}\frac{2\pi}{15} + \operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15}$

$= \left(\cot\frac{\pi}{15} - \cot\frac{2\pi}{15}\right) + \left(\cot\frac{2\pi}{15} - \cot\frac{4\pi}{15}\right) + \left(\cot\frac{4\pi}{15} - \cot\frac{8\pi}{15}\right)$
$= \cot\frac{\pi}{15} - \cot\frac{16\pi}{15} + \cot\frac{8\pi}{15} - \cot\frac{16\pi}{15}$
$= \cot\frac{\pi}{15} - \left(\cot\frac{\pi}{15}\right)$
$= 0$
$= RHS.$